Integrals: An Introduction to Research in Mathematics

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Required background. Integral Calculus (usually second semester calculus) and basic familiarity with a computational language like *Mathematica* of Maple. A basic course in number theory would be helpful, but not essential.

Description. The evaluation of integrals is one of the topics that every undergraduate calculus course covers in some detail. The goal of this project is to employ integrals as a bridge to more sophisticated mathematics. The central question to be explored is this: if somebody gives you an integral to evaluate, can you do it? If you can, what did you learn from the process and the answer? If you do not suceed, is it because it cannot be done or you have seen the right method? What does it really mean that an integral cannot be evaluated?

I hope that the examples below illustrate these points.

1) Consider the simplest of all examples:

$$\int x^n dx = \frac{x^{n+1}}{n+1} \tag{1}$$

The proof of this is elementary: it is equivalent to the fact that

$$\frac{d}{dx} x^m = m x^{m-1}$$

The proof of this simple fact will remind you of the binomial theorem, and it should inspire you to think about the definition of x^n. What does $x^{\sqrt{2}}$ mean? Or even worse, what is x^π?

The definite integral analog to (1) is

$$\int_0^1 x^n dx = \frac{1}{n+1} \tag{2}$$

Now if you differentiate (2) with respect to the parameter n you get

$$\int_0^1 x^n \ln x \, dx = \frac{-1}{(n+1)^2}. \tag{3}$$

2) During the last twenty years there has been an interesting development of *symbolic languages*. These are languages that allow you to compute a large number of integrals. Another interesting use of these languages is to create a large amount of data that will help you generate interesting conjectures. We illustrate the idea with a proof of **Wallis' formula**:

$$f(m) := \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m} = \frac{\pi}{2^{2m+1}} \frac{(2m)!}{m!m!}. \tag{4}$$

• The first approach is by **blind evaluation**. This simply means to ask the machine to do the problem for you. If you ask *Mathematica* for the value of $f(m)$ it will tell you that

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\sqrt{\pi}\,\Gamma(1/2+m)}{2\Gamma(1+m)}. \tag{5}$$

It is unclear what is going one here. What is the **gamma function** that appears in the answer? Is there a better way to find this integral? What is $\sqrt{\pi}$ doing in the answer?

If you ask for a specific then things get better:

$$f(0) = \frac{\pi}{2},\ f(1) = \frac{\pi}{4},\ f(2) = \frac{3\pi}{16},\ f(3) = \frac{5\pi}{32},\ f(4) = \frac{35\pi}{256}.$$

Now you see that the answer is a rational multiple of π, the question is how to guess the exact form of it. One the methods that will be discussed in SIMU 2002 is that of **recurrences**. For example, integration by parts will show you that

$$f(m) = \frac{2m-1}{2m} f(m-1). \tag{6}$$

From here we can guess the answer and **prove it**.

3) During SIMU 2000 we discussed the following problem. It is not hard to show that

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a) \tag{7}$$

where

$$P_m(a) = \sum_{k=0}^{m} d_k(m) a^k \tag{8}$$

is a polynomial in a with rational coefficients. A sequence of numbers is called **unimodal** if they increase up to a point and then they decrease. We had observed that $d_k(m)$ is, for fixed m, a unimodal sequence. A new proof by one the SIMU 2000 groups became a paper in the **Electronic Journal of Combinatorics**. The stronger fact that $d_k(m)$ are **log concave**: $d_k^2 \geq d_{k-1}d_{k+1}$ is still an open question.

4) Consider the iterated indefinite integrals of $\ln(1+x)$:

$$\int \ln(1+x)\,dx = (1+x)\ln(1+x) - x$$

$$\int [(1+x)\ln(1+x) - x]\,dx = \frac{x(2+x)}{2}\ln(1+x) - \frac{x(3x+2)}{2}.$$

Define $L_n(x)$ the function that appears at the n^{th} step. Then it is not hard to show that

$$L_n(x) = A_n(x) + B_n(x)\ln(1+x) \tag{9}$$

where $A_n(x)$ and $B_n(x)$ are polynomials in x. It is easy to figure out the value

$$B_n(x) = \frac{(1+x)^n}{n!}. \tag{10}$$

The value of $A_n(x)$ is very mysterious. Write

$$A_n(x) = -xD_n(x)/a_n \tag{11}$$

where $D_n(x)$ has integer coefficients and

$$b_n = \frac{a_n}{na_{n-1}}. \tag{12}$$

It turns out that

$$b_n = \begin{cases} 1 & \text{if } n \text{ is divisible by two distinct primes} \\ p & \text{if } n = p^k \text{ for some prime } p \text{ and } n \in \mathbb{N}. \end{cases} \tag{13}$$

Why?